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Calculemus 2010

17th Symposium on the Integration of Symbolic Computation and Mechanized Reasoning

Emerging Trends

David Delahaye
Renaud Rioboo (Eds.)

Cédric

Preface

This volume contains the emerging trends papers of Calculemus 2010. Calculemus 2010 was the 17th Symposium on the Integration of Symbolic Computation and Mechanised Reasoning, dedicated to the combination of computer algebra systems and automated deduction systems. As in 2008 and 2009, Calculemus 2010 was colocated with the 10th International Conference on Artificial Intelligence and Symbolic Computation (AISC 2010) and the 9th International Conference on Mathematical Knowledge Management (MKM 2010), under the umbrella of the Conferences on Intelligent Computer Mathematics (CICM 2010), organized by Renaud Rioboo and Laurence Rideau. All three conferences are thus concerned with providing intelligent computer mathematics. Although the conferences have separate communities and separate foci, there is a significant overlap of interest in building systems for intelligent computer mathematics. In 2010, the 3rd edition of CICM was held in Paris at the Conservatoire National des Arts et Métiers (CNAM) from July 5 to July 10, 2010.

There were 3 submissions. Each submission was reviewed by 2 programme committee members. The programme committee decided to accept 2 papers for publication in the emerging trends proceedings of Calculemus 2010, and to propose the last paper to be included in the proceedings of the 4th Workshop on Programming Languages for Mechanized Mathematics Systems (PLMMS 2010), which was also part of CICM 2010.

In the preparation of these proceedings and in managing the whole discussion process, Andrei Voronkov's EasyChair conference management system proved itself an excellent tool.

June 2010

David Delahaye
Renaud Rioboo

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What are the rules of "elementary algebra"?

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Abstract. Many systems dealing with mathematics, including but not limited to computer-aided assessment (CAA) software, have to deal with "equality up to the usual rules of algebra", but what this phrase means is often less clear. Even when they are clear in abstract, implementing them in a computer algebra system, which has to deal with the mathematics users have typed, complete with complications such as division (rather than multiplication by the inverse), binary subtraction etc., is far from clear. In this paper, we outline some pitfalls, and what we have learned about solving these problems.

Many systems dealing with mathematics, including but not limited to computer-aided assessment (CAA) software, have to deal with "equality up to the usual rules of algebra", but what this phrase means is often less clear, and depends on context. Our focus, as in [3], is on that variety of computer-aided assessment in which we ask "has the user got 'the right answer'?". If not, how many marks should we give, and how can we provide effective feedback, for a near miss? In order to answer the first question, we need to define equality with 'the right answer'. In this paper we identify various levels of equality, increasing in the number of objects they will consider equal (except between levels 3 and 4). To automatically provide feedback to students we often need to identify the levels between which equality fails to hold for the first time. We note that there is some ambiguity about reading what the user has entered, e.g. is `-x^b` really `-(x^b)` (normal) or `(-x)^b` (Excel)? The different decisions made during parser design mean that even experienced users may well get confused about mathematical precedence and those brought up using a variety of calculators may carry over calculator habits. Whatever the parsing precedences used, there is a lot to be said for showing the user a conventional 2-dimensional representation and asking "is this what you intended?", as STACK [11] does.

To make this paper manageable, we will consider only *natural number* coefficients, as is usual in students' work at this level. We therefore treat rational numbers as explicit quotients of integers and do not enter into questions such as "does $\frac{1}{2}$ equal `0.5`?", important though they are. We say "natural numbers" because traditional mathematical notation is ambiguous about representations

of negative integers: everyone agrees that -7 is as valid an integer as 7, and also accepts $x - 7$ as a polynomial, but, apparently inconsistently, rejects both $x7$ (which ought to be as valid as $x - 7$) and $x + -7$ (which ought to be as valid as $x + 7$). We note in passing the very subtle typographic difference in the spacing of the symbols which LaTeX uses here between $x - 7$ and $x-7$. The former is a binary minus, `x-7`, the latter juxtaposes x with the integer -7, `x{-7}`, which could be interpreted as implied multiplication.

We do not *explicitly* specify the domain of computation, as this is not usually done at this level. Where this matters it is, we claim, up to the teacher to ensure that the rules are relevant to the domain.

These "usual rules of algebra" would normally be described as 'underlying' in the terminology of [3], and therefore two expressions that differed by applications of these rules would not be regarded as 'different' from the point of view of Computer-Aided Assessment. Although we do not wish to push the analogy too far, we believe that "equality up to the usual rules of algebra" can be regarded as asking for equality at the level of some "deep structure" that underlies the "surface structure" of traditional computer-science parsing, which is the case of + and ∗ only, is given by level 6 in the next section. As [3] points out, one *might* also wish to be more relaxed about what forms of expression are to be considered as equal, and we will illustrate some opportunities for this.

One example of this is given in [7], who define[3]

$$Expr = Nat\ Integer \mid Var\ String \mid Negate\ Expr$$
$$Expr :\!+\!:\ Expr \mid Expr :\!*\!:\ Expr \mid Expr :\!-\!:\ Expr \mid Expr :\!/\!:\ Expr$$

Their context is slightly different, being focused on "interactive exercise assistants", rather than the 'marking' context of [3, 11]. We compare the two approaches in section 7.

If it were not for the *Negate*, `:-:` and `:/:` clauses, equality of terms "in the sense of elementary algebra", is precisely equality of terms modulo associativity and commutativity (but not distributivity) of `:+:` and `:*:`, and this a well-studied, and much implemented topic. Level 6 below corresponds to a canonical form for associative–commutative representations.

1   Addition and Multiplication

Let us first consider expressions built with just addition and multiplication. There are various levels of equality here. We start by considering raw strings, as typed by the student, then move to the parse trees these strings represent.

1. Textual identity. Here `x+ y` and `x+y` are considered different because of the space. Similarly `x+10` and `x+010` are different because of the leading zero[4].
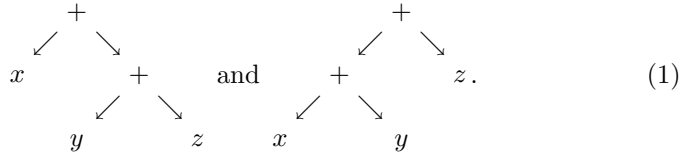
_____

[3] They use `:+:` etc. to denote the *syntactic* operators, as entered by the user.
[4] We ignore C's rule that a leading 0 implies octal!

2. Equality after lexical analysis. This would consider the above examples the same, but would distinguish `x+y*z` from `x+(y*z)` because of the extra parentheses.

3. Equality as binary parse trees. This would consider `x+y*z` and `x+(y*z)` as the same, both being the tree

$$
\begin{array}{c}
+ \\
\swarrow \quad \searrow \\
x \qquad \qquad * \qquad . \\
\swarrow \quad \searrow \\
y \qquad z
\end{array}
$$
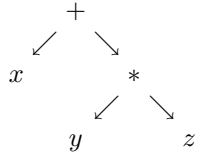
However, this interpretation would distinguish `x+(y+z)` from `(x+y)+z`, being respectively the trees

$$
\begin{array}{ccc}
+ & & + \\
\swarrow \quad \searrow & & \swarrow \quad \searrow \\
x \qquad + & \text{and} & + \qquad z\,. \\
\swarrow \searrow & & \swarrow \searrow \\
y \quad z & & x \quad y
\end{array}
\tag{1}
$$

Being based on binary parse trees, it requires an operator precedence rule (is + left-associative or right-associative) to decide which of these becomes the representation of `x+y+z`.

4. We can avoid this if we allow n-ary trees, parsing `x+y+z` as

$$
\begin{array}{c}
+ \\
\swarrow \ \downarrow \ \searrow \quad , \\
x \quad y \quad z
\end{array}
\tag{2}
$$

but this is now a third tree, different from the two trees in (1).

5. We can avoid this problem if we *flatten*[5] the associative operators + and *, which would transform both the trees in (1) into the tree in (2). While there are other ways of handling associativity, e.g. by *rotating* the second tree in (1) into the first, they do not differ fundamentally, and this is the one which generalises naturally.

6. It is customary to believe that "the rules of elementary algebra" include the facts that $+$ and $\times$ are commutative, as well as associative. In the formalism of level 5 above, this is easy to handle: we merely say that the children of a + (or *) node are *compared as (multi)sets*[6], rather than as lists, so now

---

[5] Such flattening is explicitly prohibited in some programming languages, such as Fortran [1], but then the + of floating point arithmetic is not associative.

[6] We say multisets, since `x+x+x` is different from `x+x`. In a pedagogical application, of course, we would expect to see `3x` etc., but nevertheless we must maintain the distinction. In the terminology of [3], writing `x+x+x` for `3x` would normally be considered as a *venial* error, whereas writing `x+x` for `3x` is a fundamental one.

the tree in (2) has the same children, admittedly in a different order, as the trees

$$
\begin{array}{ccc}
\begin{array}{c}
+ \\
\diagup \;\; \downarrow \;\; \diagdown \\
x \quad z \quad y
\end{array}
& \text{and} &
\begin{array}{c}
+ \\
\diagup \;\; \downarrow \;\; \diagdown \\
z \quad y \quad x
\end{array}
\end{array}
, \tag{3}
$$

and three others. We note the necessity of doing this multiset-wise comparison recursively, to ensure that

$$
\begin{array}{ccc}
\begin{array}{c}
+ \\
\diagup \qquad \diagdown \\
x \qquad\qquad * \\
\diagup \;\; \diagdown \\
y \qquad z
\end{array}
& \text{and} &
\begin{array}{c}
+ \\
\diagup \qquad \diagdown \\
* \qquad\qquad x \\
\diagup \;\; \diagdown \\
z \qquad y
\end{array}
\end{array}
\tag{4}
$$

are regarded as equal.

Beyond this we are in the realm of computer algebra, rather than syntactic manipulation. However, even for assessment of elementary expressions this move is necessary. Otherwise one is left with some kind of equality tester, often probabilistic such as [12, 13], as refined in [8], but possibly deterministic [9]. Examples of this include Maple's `testeq`. If we really want an equality test at this level, systems that act[7] *radically* in the sense of [10] compute a canonical form. Hence we define three further levels.

7. We remove identity elements: i.e. we remove 0 from the arguments of $+$ and 1 from the arguments of $\times$. Where necessary, we flatten to remove operations which end up with a single argument. For example

$$
\begin{array}{c}
+ \\
\diagup \qquad \diagdown \\
x \qquad\qquad 0
\end{array}
\;\Rightarrow\;
\begin{array}{c}
+ \\
\downarrow \\
x
\end{array}
\;\Rightarrow\; x.
$$

Pairs of $\mathcal{I}$ symbols, defined in Section 3, are removed until at most one may appear among the arguments of $\times$, and pairs of $\mathcal{R}$ symbols are removed, as discussed after (13).

8. We combine all numerical terms in the arguments of $+$ and $\times$, and (for the sake of user presentation) order the result so that any number appears first for $\times$ and last for $+$.

9. We apply the distributive law (of multiplication over addition).

Depending on the precise application, and the definition being adopted of "obviously equal", somewhere at or between levels 6 and 9 lies a system that gets many examples "right", in the sense that the user is not upset that the system refuses to recognise as equal expressions in `+` and `*` that are "obviously equal".

However, the grammar of even elementary algebra is larger than this, and includes $-$ (in both unary and binary forms) and $/$. Practically any system for

---

[7] [10] talks about "radical systems", but most systems which can behave conservatively, such as Macsyma and Reduce, have options not to.

processing mathematics has to deal with these, and their impact on the problem of "equality modulo the usual rules of elementary algebra". It might be thought that / and (binary) - posed the same problems, being the inverse operations to * and + respectively, but in fact that is not quite the case, as the unary operators behave differently.

2   Subtraction

Here we have both unary and binary -. It would seem that the informal rules of understanding mathematics make the transformation a-b→a+(-b) at a very early stage in our mental processing. Hence the expression a-b-c is thought of as a+(-b)+(-c), and hence, in levels 4 and beyond, this would become the tree

$$
\begin{array}{c}
+ \\
\swarrow \quad \downarrow \quad \searrow \\
a \qquad - \qquad - \\
\qquad \downarrow \quad \downarrow \\
\qquad b \quad c
\end{array}
\tag{5}
$$

Level 6 would then regard this as equal to the tree from a-c-b, which few would disagree with. It would, however, also regard this as equal to the expression -b+a-c. Logically, one cannot object to this, but nonetheless we "tend to prefer" one of the other forms. Convention urges minimality here, since the other forms require one fewer signs. While a notion of "simplification" can be based on minimality, as in [4], it is harder to make an argument based on "the laws of algebra".

3   Negation in Parse Trees

So far we have followed a relatively standard approach to parsing, regarding unary $-$ as a separate operator in our parse trees, even at level 9. In practice, this will not do, so we then have to decide whether -x*y represents

$$
\begin{array}{cc}
\text{-(x*y)} & \text{(-x)*y} \\
- & * \\
\downarrow & \swarrow \qquad \searrow \\
* & - \qquad\qquad y \\
\swarrow \quad \searrow & \downarrow \\
x \qquad\qquad y & x
\end{array}
\quad \text{or} \tag{6}
$$

However, the typical user of mathematics would say that these are equal, by "the usual rules of algebra", and would object to being forced to distinguish.

We therefore propose[8] replacing "unary minus" by multiplication by a special token, hereafter $\mathcal{I}$. Hence `-(x*y)`, `(-x)*y` and `x*(-y)` become the trees

$$
\begin{array}{ccc}
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
\mathcal{I} \qquad * \\
\swarrow \quad \searrow \\
x \qquad y
\end{array}
&
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
* \qquad y \text{ and } x \\
\swarrow \quad \searrow \\
\mathcal{I} \qquad x
\end{array}
&
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
\qquad * \\
\swarrow \quad \searrow \\
\mathcal{I} \qquad y
\end{array}
\end{array}
\tag{7}
$$

respectively. In the presence of flattening and commutativity of `*`, these trees are regarded as equal. What about `(-x)*(-y)`? This corresponds to the tree

$$
\begin{array}{c}
* \\
\swarrow \quad \downarrow \quad \searrow \\
* \qquad\qquad * \\
\swarrow \searrow \qquad \swarrow \searrow \\
\mathcal{I} \quad x \quad \mathcal{I} \quad y
\end{array}
\;,
\tag{8}
$$

which is equivalent to the tree from `-(-(x*y))` and many other variants with (multi)set of children $\{\mathcal{I}, \mathcal{I}, x, y\}$, but *not* equivalent to the tree from `x*y` until we get to level 7.

Algebraically $\mathcal{I}$ behaves like -1, but is distinguished from it at this level so as not to be confused with an explicit -1 entered by the user, i.e. `-x*y` versus `x*(-1)*y`:

$$
\begin{array}{ccc}
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
\mathcal{I} \qquad * \\
\swarrow \searrow \\
x \quad y
\end{array}
&
versus
&
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
x \qquad * \\
\swarrow \quad \searrow \\
* \qquad y \\
\swarrow \searrow \\
\mathcal{I} \quad 1
\end{array}
\;,
\end{array}
\tag{9}
$$

where the (multi)sets of children of a flattened multiplication operator are $\{\mathcal{I}, x, y\}$ and $\{x, \mathcal{I}, 1, y\}$ respectively.

This distinction *should*, we believe, be made at this level, where we claim that level 6 together with $\mathcal{I}$ as the interpretation of unary and binary `-` is the appropriate "deep structure". Equally though, the exercise author or equivalent may add further rules, such as levels 7, 8 and 9 to equate the two, whether with or without a penalty.

4 Division

Our treatment of division, reduction to the unary case, is similar to that of binary subtraction, even though the surface representation is different. We consider a

_____
[8] And have actually implemented in STACK [11].

unary reciprocal operator $\mathcal{R}$, and regard `x/y` as generating the tree

$$
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
x \qquad \mathcal{R} \\
\downarrow \\
y
\end{array}
\qquad . \tag{10}
$$

We need one further rule, which we will express here in a simple form, and express more generally in section 5:

"$\mathcal{R}$ distributes over multiplication", i.e.

$$
\begin{array}{c}
\mathcal{R} \\
\downarrow \\
* \\
\swarrow \quad \searrow \\
x \qquad y
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
* \\
\swarrow \quad \searrow \\
\mathcal{R} \qquad \mathcal{R} \\
\downarrow \qquad \downarrow \\
x \qquad y
\end{array}
\qquad . \tag{11}
$$

With this addition, and the rules from level 6, all the following generate the tree

$$
\begin{array}{c}
* \\
\swarrow \downarrow \searrow \\
x \quad \mathcal{R} \quad \mathcal{R} \\
\downarrow \quad \downarrow \\
y \quad z
\end{array}
\qquad : \tag{12}
$$

`(x/y)/z`, `x/y/z` and `x/(y*z)`. `x/(y/z)` generates

$$
\begin{array}{c}
* \\
\swarrow \downarrow \searrow \\
x \quad \mathcal{R} \quad \mathcal{R} \\
\downarrow \quad \downarrow \\
y \quad \mathcal{R} \\
\downarrow \\
z
\end{array}
\qquad , \tag{13}
$$

and it is a good question whether or not this should be simplified by means of $\mathcal{R}(\mathcal{R}(z)) \Rightarrow z$. Our suggestion is that it should not at level 6, but that this rule should be readily available in the toolkit of an exercise writer (e.g. at level 7).

There is one slight complication here, which is the absence of a general notation for $\mathcal{R}(z)$, at least until we allow powers and the `z^(-1)` notation. This therefore means that we should probably elide

$$
1/y \rightarrow \mathcal{R}(y) \qquad (\text{rather than } 1 * \mathcal{R}(y)). \tag{14}
$$

5 Powers

Adding exponentiation, whether just raising to an integer power, or more generally, complicates matters, and we note that it is not present in [7]. We should first note that ^ is *not* associative, and that precedence issues are therefore trickier here (as indeed they are in computer languages in general). Let us first consider the case of integer exponents. This may be either explicit integers, or (worse from our points of view) a set of problems where the integrality is implicit. This implicitness would be achieved by putting in scope rules that are not vaid in a more general context.

The first challenge is that the distributive law for exponentiation over multiplication[9] —

$$(ab)^c = a^c b^c \tag{15}$$

— seems to be viewed differently from the distributive law for multiplication over addition: we do not regard $(xy)^2$ as better or worse than $x^2 y^2$. This is easily implemented as a tree transformation:

$$\tag{16}$$

Equally, we do not regard $\left(\frac{x}{y}\right)^2$ as better or worse than $\frac{x^2}{y^2}$. The transformation (16) automatically leads to

$$\tag{17}$$

but this is not quite what was desired. It is tempting to try to "fix" this by rules taking $\mathcal{R}(x)^y$ to $\mathcal{R}(x^y)$, but our experience is that it is better to "bite the bullet" and accept that $\mathcal{R}$ is really raising to the power -1, and introduce the rules

$$\tag{18}$$

_____

[9] The other distributive law, $a^{b+c} = a^b a^c$, *does* seem to be in the same camp as the distributive law for multiplication over addition, and should probably not generally be applied.

`1/x^(-2)` is then represented as $x^{\mathcal{I}*\mathcal{I}*2}$. It is probably a matter of taste whether one regards `1/x^(-2)` as a clumsy expression which does not deserve to be simplified, or whether one wishes to work at level 7, in which case $\mathcal{I}*\mathcal{I}*z \Rightarrow z$ (or the pair $\mathcal{I}*\mathcal{I} \Rightarrow 1$ and $1*z \Rightarrow z$) need to be added to the ruleset.

6 Powers: non-integral exponents

We note that these re-write rules for exponentiation are subtle once we get to non-integral exponents and that even great thinkers have made mistakes, such as [5]

§148. Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$ we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$ or 2, for the value of the product of $\sqrt{-1}$ and $\sqrt{-4}$. Thus we see that two imaginary numbers, multiplied together, produce a real, or possible one.
But, on the contrary, a possible number, multiplied by an impossible number, gives always an imaginary product: thus $\sqrt{-3}$ by $\sqrt{+5}$, gives $\sqrt{-15}$.
§149. It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{\frac{a}{b}}$, it is evident that $\sqrt{-4}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$ or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided by $\sqrt{-1}$ gives $\sqrt{\frac{+1}{-1}}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$.

A more detailed discussion of the mathematics underlying these issues is given by [6]. When it comes to expressions, rather than numbers, (15) is not in general valid: compare
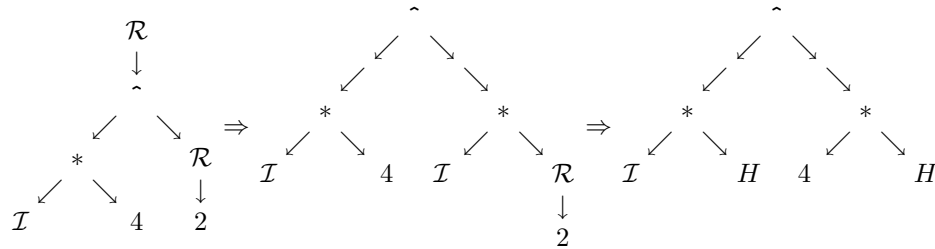
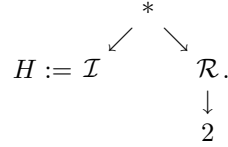$$\sqrt{1-z}\sqrt{1+z} \overset{?}{=} \sqrt{1-z^2} \tag{19}$$

with

$$\sqrt{z-1}\sqrt{z+1} \overset{?}{=} \sqrt{z^2-1}, \tag{20}$$

where (19) *is* universally valid, but (20) only on a halfplane. These difficulties are discussed in [2], where the proofs of validity are fundamentally not of a rewriting nature.

As an example, consider $\frac{1}{\sqrt{-4}}$: we can operate on this parse tree using (18) then (16) to give

9

(i.e. $(-)^{-\frac{1}{2}} \times 4^{-\frac{1}{2}}$) where $H = -\frac{1}{2}$ represents the tree

$$H := \mathcal{I} \overset{*}{\underset{\mathcal{R} \atop \downarrow \atop 2}{\diagup \diagdown}}.$$

Further computer algebra rules are required to decide how $(-)^{-\frac{1}{2}} \times 4^{-\frac{1}{2}}$ should be dealt with.

7   Comparison with [7]

The paper [7] cites five basic assumptions, which we reproduce below (our layout), and comment on (marked **C**).

1a Associativity of operators is implicit, meaning that a user cannot and should not distinguish a+(b+c) from(a+b)+c. The system can thus minimize the use of parentheses in presenting terms.

C Level 5 and later deal with this.

1b Commutativity, on the other hand, should be used with care. We want to respect the order in which terms appear as much as possible for a better user experience.[10]

C This can be achieved by not aggressively re-ordering the multiset of children of a **+** or ***** node.

2 Constant terms are normalized aggressively: the skills to manipulate fractions and integers are assumed to be present.

C This contrasts with our approach, where we would normally prefer[11] the answer **4** over **2+2**. However, adding constant normalisation rules as "underlying" rather than "venial" would achieve this goal.

3a The distribution of multiplication over addition (law [M5]) is an explicit step in the derivation.

C This is essentially the difference between our level 6, which we claim to be the appropriate "deep structure", and level 9.

3b Laws to manipulate the sign of a term (laws [N1] up to [N6]) can be performed automatically.

[**N1**] $-(-a) = a$
[**N2**] $a - a = 0$
[**N3**] $a - b = a + (-b)$
[**N4**] $-(a + b) = (-a) + (-b)$
[**N5**] $-(a \cdot b) = (-a) \cdot b$
[**N6**] $-(a/b) = (-a)/b$

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[10] Recall that they are *generating* terms, whereas we are not.
[11] In the sense that we would mark **2+2** down.

C [N3], [N5] and [N6] are consequences of our treatment of $\mathcal{I}$: the others could certainly be added as underlying rules. [N4], in particular, is the equivalent for $\mathcal{I}$ of (11) for $\mathcal{R}$.

It would therefore be reasonable to state that our level 6, augmented by $\mathcal{I}$ and $\mathcal{R}$, and (11), provides a subset of the functionality [7] need, and one which can be further enhanced to theirs by adding certain rules as 'underlying' — rules that we would probably wish to have as "venial' in any case.

8   Conclusions

We claim that "equality up to the usual rules of algebra", for the four operations of `+`, `-`, `*` and `/`, can be represented as equality of the deep structures of level 6 (associative–commutative equality for `+` and `*`) together with $\mathcal{I}$ and $\mathcal{R}$ for unary subtraction and multiplicative inverse. Further rules, such as those posited in [7], can then be added on top of these if required, but these rules alone suffice for the purposes of much computer-aided assessment.

If one wishes to add `^`, things become more complicated. They become more complicated pedagogically because they *are* more complicated mathematically, principally because fractional powers require multi-valued inverses. This moves us away from an *algebra* of simple expressions represented by parse trees. Restriction to explicit integer exponents can be dealt with by the mechanisms we have proposed.

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Interactive Documents as Interfaces to Computer Algebra Systems: JOBAD and Wolfram|Alpha*

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Abstract. Interactivity and customization are common trends guiding the design of services on the web. Not only can users adapt content to their preferences, they can also dynamically aggregate content from various sources on interactive pages in their browser that thus turn into powerful command centers (e. g. iGoogle). Our JOBAD architecture embeds mathematical services into XHTML+MathML documents. JOBAD is a modular JavaScript framework for interactive services such as term folding or definition lookup.

We have now enhanced it with a client for computer algebra. It lets the user select mathematical expressions and ask a CAS to compute, graph, or rewrite them. We have done first steps towards an integration with the Wolfram|Alpha web service API, which gives access to Mathematica as well as a large mathematical knowledge base. We are currently working on a generalization towards arbitrary CAS backends and thus promoting documents as interfaces to computer algebra systems.

Keywords: Interactive Documents, Computer Algebra Systems, Web Services, MathML, OpenMath

1 Introduction

Writing and reading mathematics is difficult. Due to its abstract and succinct notation and the practice of omitting information that can be inferred from the context, readers — especially if not experts on the subject matter — often lack knowledge about certain characteristics and properties of the objects under consideration. Moreover, due to the hierarchic nature of mathematical theories, many of the terms defined and used in a mathematical document actually depend on other terms defined elsewhere.

From another point of view, a current development of the Internet is to provide ever more possibilities for the users to receive data from different sources, thus overwhelming the user with useless information. We try to cope with this by setting up intelligent contextual filters that act as interfaces to the raw data.

The point where the above two perspectives meet is the JOBAD architecture [13, 9]. Its goal is to facilitate the integration of diverse web services into mathematical documents — inspired by the Web 2.0 technology of *mashups* [23, 2]. Our vision of an *interactive document* is a document that the user can not just read, but adapt according to his preferences and interests *while* reading it — not only by customizing the display of the rendered document in the browser, but also by changing notations (which requires re-rendering) or retrieving additional information from services on the web. Consider a student reading lecture notes: Whenever he is not familiar with a mathematical symbol occurring in some formula, JOBAD enables him to look up its definition without opening another document, but right in his current reading context. Or consider the problem of converting between physical units (e. g., imperial vs. SI). Instead of manually opening a unit converter website and copying numbers into its entry form, we have enabled an in-place conversion.

JOBAD handles enriched mathematical documents presented on the web, as a combination of XHTML, RDFa, MathML [28] (for both display and semantics via Presentation and Content MathML) and OpenMath [3] (as an alternative language to Content MathML in which the semantic structure of the formulae is annotated) that are (inter)active and, therefore, customizable. For example, JOBAD provides services for the user that change the display and the content of the active document by retrieving additional data from different web services.

We have used JOBAD to enrich the General Computer Science lecture notes at Jacobs University which are written using sTEX(semantically enhanced TEX), uploaded to an installation of TNTBase, a versioned XML database [31, 27], which translates them to OMDoc, and finally to XHTML+MathML+RDFa [7]. The architecture is not constrained to that and can also be used for other purposes. For example, the LATIN project (Logic ATlas and INtegrator) [17], which aims to use a "logics as theories/translations as morphisms" approach to achieve the interoperability of both system behavior and represented knowledge (the Logic Integrator), and to obtain a comprehensive and interconnected network of formalizations of logics of computational logic systems (the Logic Atlas); see [6] for the details of the JOBAD integration. Another example in this area can be given from history: in the beginnings of the 20th century, a group of (mainly) French mathematicians wrote the basis of set theory and published under the common pseudonym: Nicolas Bourbaki. But, for this collection of books, there is no digitized version which would allow the users to explore (e. g. the basis of set theory) properly.

The JOBAD architecture is modular and easily extensible, which gives other developers the ability to develop customized service modules for different tasks. This is where Computer Algebra Systems (CASs) come into the picture. In this work we present a new JOBAD service that can interact with CASs. Our generic service is instantiated to connect to Wolfram|Alpha using the Wolfram|Alpha web service API. Thus JOBAD can provide a lot of background information for a term or an equation, e. g., we can use it to simplify the selected term, plot it, or compute solutions to equations.

2 Related Work

Similarly to JOBAD, the ActiveMath project [1] deals with *aggregated documents* which are retrieved from a knowledge base depending on the user's topics of interest (what the user wants to learn) and its prerequisites. It is presented as a platform for learning mathematics in school and university.

MathDox [5] is an XML based format for interactive mathematical documents which can be transformed to interactive mathematical web pages using the MathDox Player. MathDox uses OpenMath for semantic representation and was actually designed, in part, to interact with CASs like Mathematica, Maxima and GAP via OpenMath phrasebooks (cf. [3]), so this can be considered as related work regarding our project. For MathDox, in order to evaluate a certain mathematical formula in a CAS, several steps need to be taken by the MathDox player for transforming the underlying formats of the document (DocBook, OpenMath, MONET etc.) to HTML which embeds the result of the CAS query.

Interactive exercises have been developed by both ActiveMath [10] and MathDox which rely on a user's answer and a solution checker in order to return feedback to the user. In order to evaluate the user input, ActiveMath needs a CAS to check for correctness and relies on one of the following: Yacas, Wiris-CAS or Maxima. Still, the features provided by the JOBAD architecture are more inclined towards modularity and client-side services that neither presuppose a single backend nor a particularly powerful one, whereas the two aforementioned projects are less modular (we are refering here at the addition of new services by third party sources, which JOBAD can handle very well) and more of the required computation is done on the server instead of the client.

3 Computer Algebra Services for JOBAD

The research in the field of CAS has received mass recognition in May 2009 when Wolfram|Alpha [30], a computational knowledge engine, was launched. Wolfram|Alpha is based on two primary resources for the answers it provides (as it is classified as being an "answer engine"), the Mathematica backend and the knowledge base. Mathematica [20], whose 7th version was released in the first quarter of 2009, is a well-known CAS, which provides many possibilities in interacting with mathematical formulae. From a mathematical point of view, the way Wolfram|Alpha works is that it always tries to return everything that it knows about a certain formula (factorization, roots, plot) or already knows about a certain formula (as one can see in Figure 1).

An integration of the services provided by Wolfram|Alpha with the JOBAD architecture makes sense, as this would facilitate the users' immediate access to more information regarding the formulae that the user explores in a document, thus providing, besides the already existing information services (e. g. definition lookup), another way of acquiring background information regarding the topic, thus making it easier for users to understand complex mathematical formulae. This data is instantly computable and is available for access via

the Wolfram|Alpha website or via a webservice API specially designed for developers. Still, Wolfram|Alpha is only one example of a CAS, and the JOBAD architecture should not be confined to only using this one. Another example of a similar system with which JOBAD could interact are those CASs that deal with OpenMath content and which can be reached via the SCSCP protocol [12].

The work envisioned with this project has two main parts that, in turn, regard the improvement and extension of the already existing JOBAD architecture [9]. The idea is to develop another module for the already existent JOBAD architecture that will allow the user to interact with more web-driven mathematics, via the Wolfram|Alpha computational knowledge engine. The extension also regards a generalized method of a "Send To" menu that will allow the user to select an annotated MathML fragment (formula) and redirect it to some other sources of information, in this case, a CAS, in particular, Wolfram|Alpha. Wolfram|Alpha was chosen as an initiator for the "Send To" method, as this seems the most useful, rational and complex choice for a user who wants to look up mathematical content on the web, as Wolfram|Alpha is also capable of plotting different functions, identifying equations, terms etc. Still, this would only be the first use case for the menu, as this can be further expanded and further destinations for the "Send To" menu can be provided. This new extension, in theory, should work with any CAS, the only constraint being the CAS and the mathematical content in the document should have a common language — such as OpenMath —, or that there should be a one-to-one mapping between these languages and that there is a way to specify the desired operation to be performed by the CAS on the user interface. The difference between Wolfram|Alpha and other CASs is that Wolfram|Alpha will automatically return plots, derivatives, related formulas, while these, if supported, have to be explicitly asked for in the other CASs. This functionality needs to be embedded in the respective service GUI elements and will have to be adjusted per CAS. In Figure 2 you can see an updated diagram of the entire JOBAD architecture, with the components in red being the ones to be added (also, the proxy will be redesigned).

Another improvement of the architecture comprises modifications to the user interface that, right after the document has been loaded and JOBAD will come in to place, it will display a notification on the page that will allow the user to select the necessary and wanted services available for the respective document. Then, the data will be stored for later access so that each and every time the user will display the document again, the offered services are seamlessly loaded and the document is prepared for interaction according to the user's preferences.

4   Wolfram|Alpha in JOBAD

The steps required for the extension of JOBAD are related to two important tasks, first, querying the CAS system (in this case the Wolfram|Alpha engine) and as for the improvement of the user interaction, retrieving the results from the system and displaying them for the user.
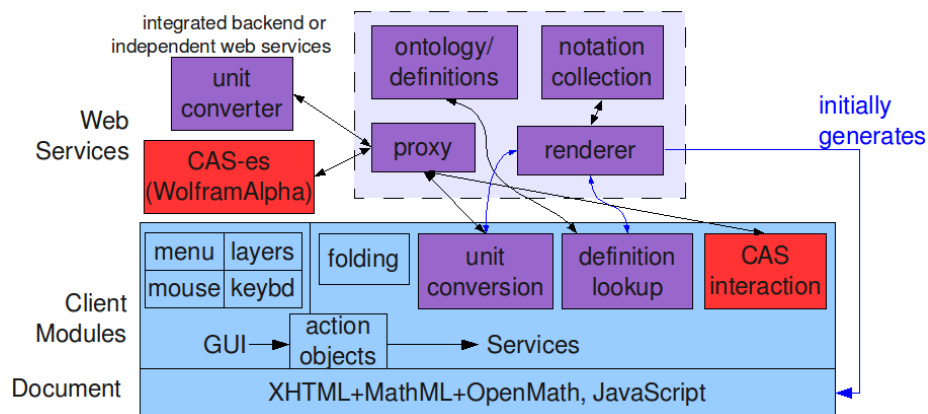
Fig. 1. Wolfram|Alpha results for "subset"

Fig. 2. JOBAD architecture

4.1 Querying Wolfram|Alpha

In the initial phase of the project, there were two ideas around which the retrieving of data revolved. First of all, there was the brute force way, query the Wolfram|Alpha website, wait for it to load (as it contains a lot of JavaScript and AJAX requests) and retrieve the desired content from the webpage via XPath or other means of accessing XML fragments and display it in the associated dialog box. Still, there were some issues regarding this:

– The Wolfram|Alpha website only provides instant results image-wise. The images are generated on-the-fly and then deleted shortly after the AJAX request has been completed; therefore, there was no way to return the images which are rather important in the case of functions.
– The results were displayed via JavaScript and AJAX, which means that retrieving the loaded page through our proxy (which is needed to circumvent security restrictions; see below) would be hard, if not impossible.
– One can not set the content of the retrieved results which is, by default images, even for mathematical formulae.
– Further content cannot be retrieved or filtered (we are interested in retrieving the meaning of the formulas, not just the graphical representation)

The Wolfram|Alpha service provides a web-based API for clients to integrate the computational and presentation capabilities of Wolfram|Alpha into their own applications or web sites. The Wolfram|Alpha webservice allows one to query the database as if one were to query the actual website, but allowing for specifying the type of desired information more exactly, and providing additional functionality. As the above solution seems infeasible for an automated client, we have applied for and obtained a grant for research purposes on the Wolfram|Alpha architecture, which consists of a Wolfram|Alpha API key. Regarding the extra functionality provided by the API, the output of the Wolfram|Alpha webservice

can be filtered according to the developer's wish in order to better accommodate the needs of the user. The filtering regards the various possible representations of the result, providing options for visual representations (e. g. images, HTML, PDF) or textual representation (e. g. plain text, Mathematica syntax, an XML encoding of the Mathematica syntax called ExpressionML [8] etc.). Once there is one possible way to query the Wolfram|Alpha engine, the only problem that still remains is establishing a common language for the interaction between our document and Wolfram|Alpha.

As Wolfram|Alpha does not support querying mathematical content via semantically enriched MathML (Content MathML) or OpenMath, one is required to query the engine via other mechanisms, this being the first important step towards querying the knowledge base. The common syntax for querying seems to be the Mathematica language which is familiar to the Wolfram|Alpha engine, as it relies on a Mathematica backend, therefore requiring a translation from the existing Content MathML and OpenMath standard served by the server (in our case TNTBase) to Mathematica. We are aware of existing transformations between OpenMath and Mathematica, e. g. on the server as a part of the MathDox infrastructure [18, 5], or on the client as a part of the Sentido formula editor [11], and have decided to use the MathDox infrastructure for translation.

Also, taking into consideration the limitations of JavaScript, we will need to set up a proxy in order to access Wolfram|Alpha, as JavaScript code is not allowed to provide data unless it comes from the same domain ("Same Origin Policy") [21]. The purpose of the proxy will be to only prepare a request for forwarding to the Wolfram|Alpha server and, when the data is available, to provide it back to the JavaScript client, given that the key that we received from Wolfram|Alpha may not be exposed and will be stored in the proxy. Also, depending on the nature of the data, the proxy might alter the structure of the retrieved document in order to save post-processing on the client side. The purpose of the proxy is to interact with the "outside world" of the application, retrieve necessary content, fit it together nicely and then return it to the user. Given the choice of different formats in which Wolfram|Alpha can return the result, there are two possible ways of presenting the information retrieved from Wolfram|Alpha on the client: embedding it as parallel presentation and content markup (i. e. Presentation MathML annotated with Content MathML or OpenMath) into the original document, thus effectively rewriting formulae in the document, or simply displaying it for the user's information.

The communication flow required for rewriting expressions is vizualized in Figure 3. The test case for the integration of Wolfram|Alpha services into JOBAD was the previously mentioned lecture notes which are usually displayed in MathML (Presentation) and also have content annotations in OpenMath. As the annotations are made up to a per symbol level, it is easy for the Wolfram|Alpha service of the JOBAD architecture to find the associated OpenMath representation of the selected expression and make an HTTP POST request to the proxy which runs on the same domain and port (due to the "Same Origin Policy"). The proxy will then determine if the content is OpenMath and, in this case, will send a
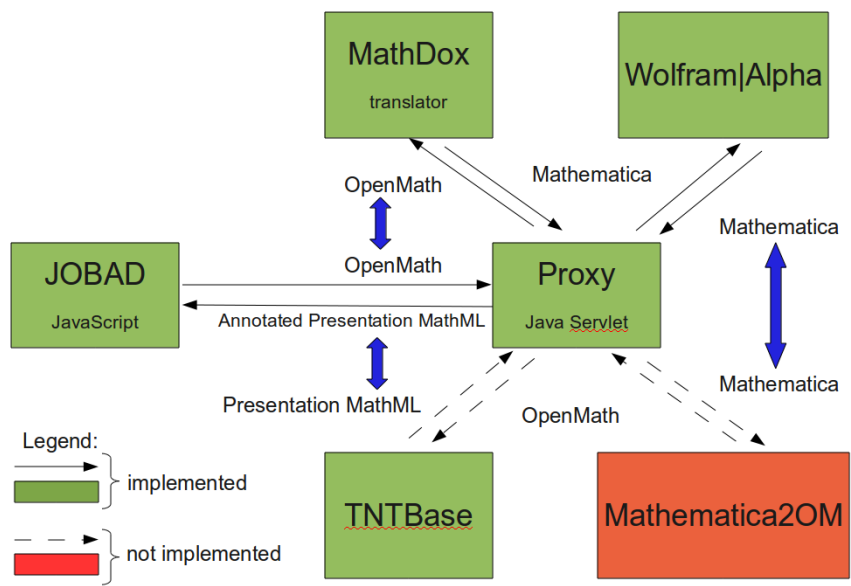
Fig. 3. Proxy architecture for content-math oriented tasks

request to a webservice running on the MathDox [19] website which will translate the OpenMath content to a Mathematica expression. As Wolfram|Alpha is based on Mathematica (the plots, expansions, etc., are computed via Mathematica), the Mathematica language is easier to understand by the engine and the computed results are more relevant to the search, as no Natural Language Processing (NLP) techniques need to be employed to transform the input (e. g. on a basic level, an input as "Sqrt[x]" might produce more relevant results than "square root of x"; for this simple test case, the results are identical, but for more complex queries, NLP processing might not work). So, the converted OpenMath expression is then passed to Wolfram|Alpha for evaluation in two steps: the first request is for Mathematica output (a representation of the formula in Mathematica language) and is directed towards the content and meaning of the formula, while the second request is sent in order to retrieve pictures and a Presentation MathML representation of the results.

Given that the first query was successful (which can be easily verified in the result of Wolfram|Alpha query), the system should proceed in transforming the retrieved Mathematica content to a displayable form (Presentation MathML), while still preserving the associated content annotation. For this, the following possibilities have been investigated:

- *NB2OMDoc*: Developed by Klaus Sutner NB2OMDoc [26] is a Mathematica package that is able to transform Mathematica code (version 4.2, latest version is 7) to OMDoc (an early draft of version 1.2). The disadvantages of this system would be that it requires Mathematica to be installed on the proxy computer and that it is designed for an old format of both Mathematica and OMDoc. In adition to that, one would have to transform (render) the OMDoc content to Presentation MathML with OpenMath annotations, a step to be executed by TNTBase and its embedded renderer from the JOMDoc library [14], a Java API for OMDoc documents (and illustrated in the picture).
- *Mathematica web service*: As pointed out here in the Mathematical online tutorial[1], Mathematica is capable of exporting its formulas to both Content and Presentation MathML. So, one can design a web service that would start Mathematica, input a formula, convert it to MathML and then retrieve the result. This is not feasible, as the Mathematica files (with extension nb) have a proprietary format and extracting content from that file is not easy. Also, another drawback is that one would have to start Mathematica each time (as we are not aware of a Mathematica daemon) which, even on a new computer, takes more than 10 seconds which makes a webservice not user friendly. An example in this area is WITM [29], Web Interface to Mathematica which provides a Mathematica interaction inside the browser. Still, the main constraint is that WITM (and similar attempts) is intended to allow a small number of licenced users access to Mathematica kernels remotely, but

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[1] http://reference.wolfram.com/mathematica/XML/tutorial/MathML.html

not simultaneously (a large number of users might mean interference in the result)

– *Sentido formula editor*: Developed by Alberto González Palomo as part of the Sentido [25] editor, browser and environment for OMDoc, it is a JavaScript extension that allows the translation between different mathematical formulae representation formats. This would mean that all the translation between the different formats (Mathematica to OMDoc) should be done on the client side. The drawbacks of using this method is that the entire library is necessary for this and there seems to be no interface to just transform between the different formats, bypassing the other functionality.

Since each of the methods presented above has its own (major) drawbacks and would require more time to integrate, we consider the integration of a Mathematica to OpenMath/OMDoc/MathML translator as future work.

4.2   Displaying the results

Once the data is retrieved in a displayable format (images, Presentation MathML, rendered MathML from Mathematica), it needs to be displayed. Following the design pattern used before in the definition lookup service, we decided to use the same jQuery UI [15] widget that allows the developer to populate a dialog window with content, in this case the results provided by Wolfram|Alpha, translated from their XML representation to XHTML. The expansion is made in place, where the user clicked and allows the user to move the dialog around (examples can be viewed in Figures 4 for definition lookup, 5 for the module loading utility and 8 for the Wolfram|Alpha lookup). An alternative that is appropriate for some types of queries and results is rewriting mathematical expressions in place, which JOBAD so far does for the results of unit conversions [9].

4.3   Preserving Document Settings

The last part of this project regarded the extension of the interface with another service that allows the dynamic loading of other services, thus providing even more freedom of configuration on the user side, leading towards more personalized active mathematical documents.

This extension first adds a text at the top of the document ("Click me to configure the loaded modules") and uses the same jQuery UI dialog (as one can see in Figure 5), only that this time, the dialog is made modal: everything else except the dialog is grayed out and it does not allow access to the underlying document until either the form is confirmed (via the *Ok* button) and the necessary modules are loaded or the dialog is closed via the **x** button. In addition to that, we imagine students that might access a document or documents on the same domain for multiple times and having to load the same modules over and over might become irritating and annoying. Therefore, in addition to loading the necessary services, this module also stores the loaded services in a cookie for further usage and each time a page is loaded, the cookie is retrieved and the

DEFINITION:

If one shape can become another using turns, flips and slides, then the two shapes are said to be congruent.
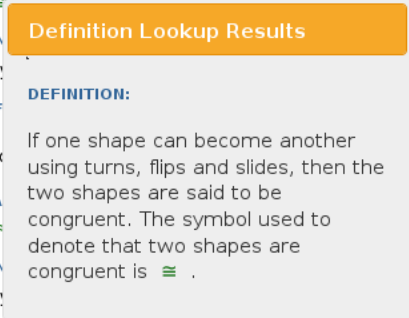
EXAMPLE:

$F \cong$

CON
A sy

DEF

Tw

EXA
$F \cong$

CON
A sy

DEFINITION:

Definition Lookup Results

DEFINITION:

If one shape can become another using turns, flips and slides, then the two shapes are said to be congruent. The symbol used to denote that two shapes are congruent is $\cong$.

...n them is their size, and possibly the need to turn or flip ...

Fig. 4. Definition Lookup Example

modules that have been loaded at the last access of the web page are loaded again. The list of available services is not static, but rather dynamic and each time the top of the page is clicked, a request is sent to the server, asking for the available services. This can be further expanded in a server-side saved user profile that can be used in other projects as the adaptive document browser panta rhei [24, 22].

5   Example test case

In the following section we present an example workflow for a test document. The user arrives at the test document and no services (besides the service loading system) are loaded, resulting in no obvious functionality. Once the user clicks the top of the page text which allows the loading of additional modules, a request is sent to the server asking for the available services, the dialog is populated and pops up and allows the user to check the *wolframalpha* checkbox (see Figure 5). Once both the *wolframalpha* and the additional *folding* services are loaded, the user proceeds to the document and after each right click is presented with a contextual menu, dynamically created for that document element. Assuming the user would right click on a mathematical fragment which is $\sqrt{x}$, with the associated XHTML fragment presented in Figure 7 which contains both Presentation MathML and annotations in OpenMath format, he would then receive a visual confirmation of his action via a context menu, as one can see in Figure 6. If a user were to access the Wolfram|Alpha website and search for the Mathematica representation of the OpenMath fragment, in this case *Sqrt[x]*, the result page would look like Figure 9. After the request is processed, the Wolfram|Alpha content is retrieved on the client side and the user will experience something resembling Figure 8.
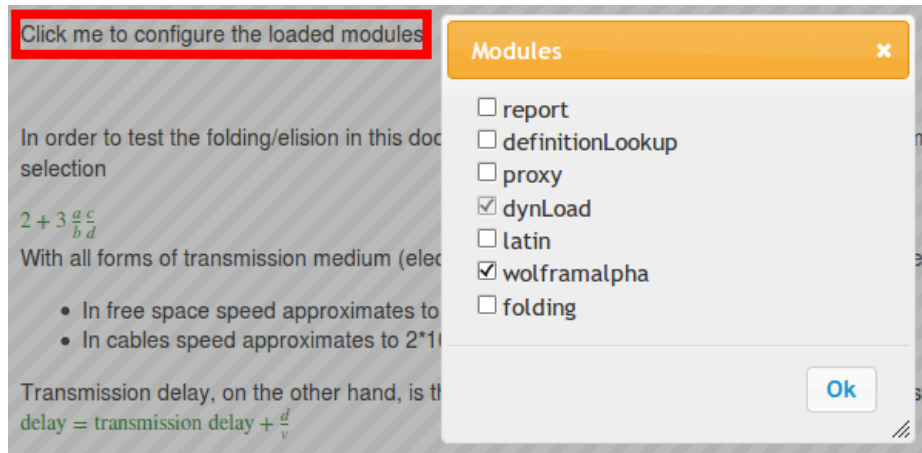
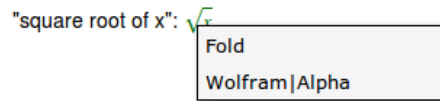**Fig. 5.** The user loads the *wolframalpha* service

Fig. 6. The user performs a right on the $\sqrt{x}$ symbol

Fig. 7. The Presentation MathML and the associated OpenMath content annotation representations of $\sqrt{x}$
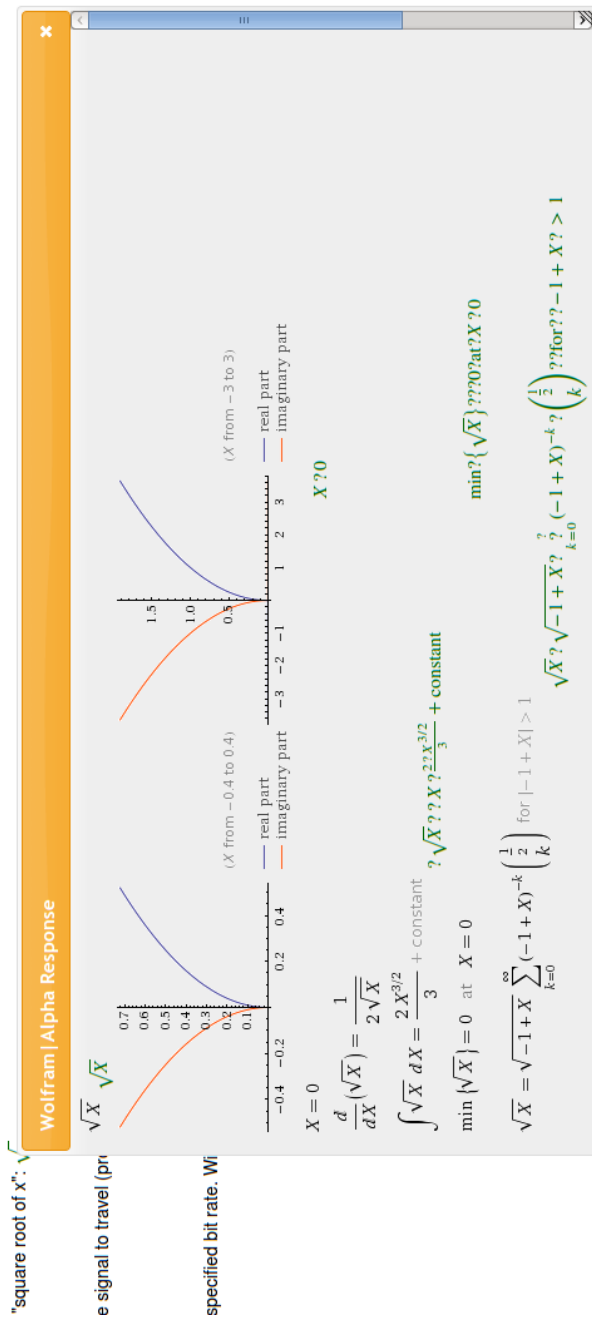
Fig. 8. Part of the Wolfram|Alpha results embedded into the original document (The question marks in the MathML formulae result from Wolfram|Alpha using a wrong character encoding; we are working on a workaround for that.)
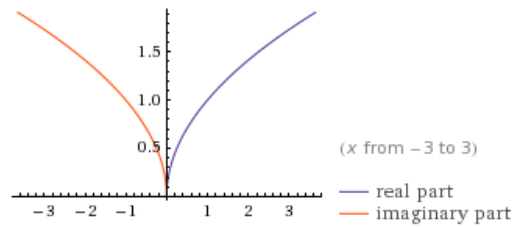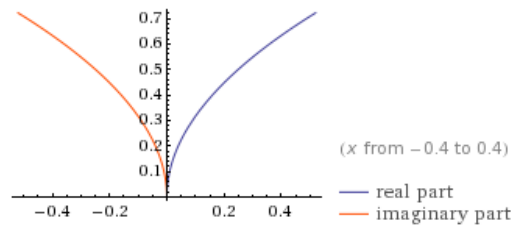
Fig. 9. Part of the Wolfram|Alpha website search results for *Sqrt[x]*

5.1 Extending the Wolfram|Alpha interface to other CASs

This section of the project represents more of a further research topic. While Wolfram|Alpha is very competent, the amount of information it provides might not be adequate for other services or facilities or might not be detailed enough, thus requiring a more involved or specialized CAS. Some things also to keep in mind are the efficiency of the CAS, as this will have a great impact on the users, and the licensing of the CAS (the Wolfram|Alpha API is not freely accessible and requires licensing). Therefore, instead of using Wolfram|Alpha for simple or very specific computations, one can use a locally installed CAS or just another CAS that handles things better and/or faster than the webservice provided for Mathematica. Still, as presented before, Wolfram|Alpha is only a specialized application of JOBAD's "CAS" service, as this feature could be implemented for any existing webservice, given that there is a common interaction language between the MathML/OpenMath formulae in the documents and the CAS. However, given that Wolfram|Alpha can accomplish much more than a simple CAS, as it also relies on an index for words and that its output is a bit different from the one of a CAS (which is punctual), a transition from Wolfram|Alpha to a simple CAS might be a bit more involved and will require some further investigations both in terms of CASs and in terms of integration with the JOBAD architecture.

6 Conclusion and Future Work

We have presented the design and first implementation steps for an integration of CAS services into the JOBAD architecture for interactive mathematical documents. Specifically, we provide a service that lets users interact with the Wolfram|Alpha web service API. Its final version will permit users to send symbols or entire annotated mathematical formulae for evaluation to a CAS, the result of which will be displayed contextually.

We will evaluate the CAS module for JOBAD using the existing integration of JOBAD with the first-year undergraduate computer science course taught at Jacobs University, whose slides and lecture notes are written using the LaTeX package sTeX. sTeX documents are automatically converted into OMDoc [16] documents and from there into JOBAD-enabled XHTML+presentation/content-MathML documents [7]. Thus, students will be able to interact with Wolfram|Alpha-generated content.

In future work, we will extend the architecture to other CASs. This will also act as a simple integration platform between CASs as results received from one system can be sent to another one. Similarly, the current design can be extended to theorem provers where, instead of querying the service for the simplification of an expression, we query for a proof of a theorem. This could be the first step to a document-centric integration of theorem provers both among each other and with CASs.

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